PROPAGATION OF TRANSIENT ACOUSTIC WAVES IN LOSSY AND LOSSLESS MEDIA

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ABSTRACT

This paper presents a computationally efficient technique for calculating the transient wave from a planar source within a medium. The spatial and temporal excitation are known at the input plane. The technique is applicable to lossless media, to media with a loss coefficient that is linear in frequency, and to media with a loss coefficient that is quadratic in frequency. The technique is computationally efficient in that it relies on FFT algorithms for the calculation rather than integral solutions requiring geometrical interpretation. In this method, we find the Green's function that solves the applicable wave equation and that meets the required boundary conditions in the source plane. This Green's function is then used in a form of the Kirchhoff integral that applies to transient wave propagation and we find the response to a time-domain impulse excitation. The solution is then expressed in the spatial frequency domain where a linear systems interpretation provides a physically intuitive interpretation of the results. The propagation is seen to be represent a time-varying spatial filter that increasingly attenuates the higher spatial frequencies as time goes on. Unlike the continuous wave case, the filter is neither band-limited nor a pure phase filter. The particular form of the spatial filter depends on the medium assumed and on the baffle conditions. The solutions for the impedance-matched baffle and the resilient baffle can be expressed in terms of the solution for the rigid baffle case. Several examples of calculated fields will be given.

INTRODUCTION

The problem that we wish to solve can be stated using the geometry of Fig. 1. Given the z-directed velocity excitation over an arbitrary shaped region of the z=0 plane, we wish to find the acoustic velocity potential $\phi(x,y,z,t)$ at an arbitrary point in the positive-z half-space. The region in the input plane will be assumed to be rigidly baffled. (It has been shown [1] to be possible to relate impedance-matched boundary conditions and resilient boundary conditions to the solutions for the rigid baffle.) We will assume that the time and space variations of the input z-velocity are separable and that the z-velocity is given by

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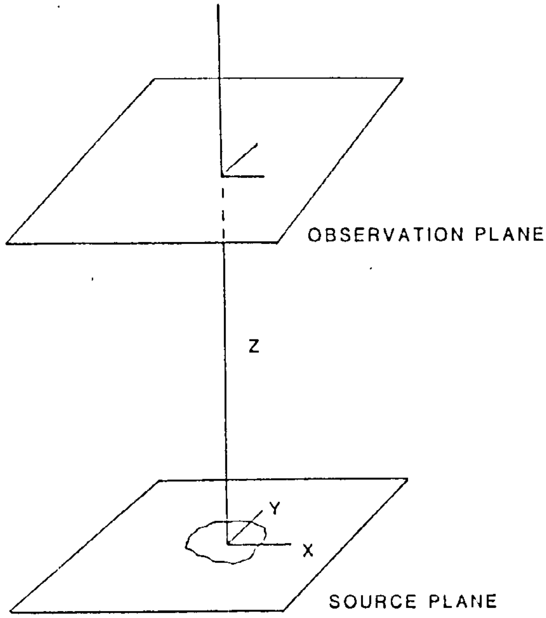

Fig. 1 Source and receiver geometry

$$v_z(x,y,0,t) = T(t)s(x,y) \tag{1}$$

The method will be based on the spatial impulse response technique used by Stepanishen[2-6] and reviewed by Harris[7]. In this technique, it is shown that the relation between the acoustic potential and the input z-velocity is

$$\phi(x,y,z,t) = T(t) \underset{t}{*} h(x,y,z,t) \tag{2}$$

where h(x,y,z,t) is known as the **spatial impulse response** in similarity to approaches used in linear systems theory. The * symbol indicates convolution over the variable appearing immediately below it. The spatial impulse response is defined as the velocity that will result when the source is excited by a z-velocity of the form $s(x,y)\delta(t)$ where $\delta(t)$ is the Dirac impulse function. Hence the problem is reduced to one of finding the spatial impulse response of the assumed spatial excitation.

Other approaches have been successful in computing the desired potential and are reviewed in Ref. 7. More recent techniques include Refs. 8-10 and 11.

In this paper we will seek to find the potential for lossless media, for media with a linear frequency dependence of its attenuation coefficient (over a portion of its frequency response), and for media with a quadratic frequency dependence of its attenuation coefficient (again, over a portion of its frequency range). The application of the impulse response technique to the latter two cases is a new contribution of this paper. An additional feature of this paper is the representation of the solutions in a form that is readily computed by FFT methods (or fast Hankel transform algorithms for axisymmetric cases) without resorting to geometrical interpretations of the integrals.

The technique for each medium will begin with a representation of the wave equation model of that medium. The Green's function (or the two-dimensional spatial transform of the Green's function) that solves the wave equation and satisfies the assumed rigid baffle boundary conditions will be displayed. From this Green's function (or transform), the potential at the observation point (or its two-dimensional spatial transform) will be related to the input z-velocity (or its spatial transform). Space restrictions do not allow for the derivation of all results but complete derivations are in preparation for submission for publication[12,13].

CASE I: LOSSLESS MEDIA

The wave equation is the Helmholtz wave equation,

$$\nabla^2\phi - \frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} = 0 \tag{3}$$

The Green's function for the rigid baffle is known to be (assuming only outward travelling waves),

$$g(x,y,z,t) = \frac{\delta(ct-R)}{2\pi R} \tag{4}$$

where

$$R = (x^2 + y^2 + z^2)^{1/2} \tag{5}$$

The spatial impulse response of this problem is[12]

$$h(x,y,z,t) = 2s(x,y)\underset{xy}{**}\delta[t-(R/c)]/2\pi R \tag{6}$$

Since the spatial convolutions are simplified in the spatial transform domain, we take the spatial transform of Eq. (6) to get

$$\bar{h} = (1/\pi)\bar{s}J_0[\rho(c^2t^2-z^2)^{1/2}]H(ct-z) \tag{7}$$

where the overbar indicates the two-dimensional spatial transform and $\rho=(f_x^2+f_y^2)^{1/2}$. In this form we can identify the $J_0[\rho(c^2t^2-z^2)^{1/2}]H(ct-z)$ term as a time-varying spatial filter for the propagation in lossless media from a source in a rigid baffle. Since the results are going to be computer-implemented and normalized to maximum values, we will drop the multiplicative constants. Calling this spatial filter the **propagation transfer function** $\bar{h}_{p1}$ we have

$$\bar{h}_{p1} = J_0[\rho(c^2t^2-z^2)^{1/2}]H(ct-z) \tag{8}$$

We find the spatial impulse response for a given value of z in the following way. We calculate the spatial transform of the given s(x,y) function, calculate the values of $\bar{h}_{p1}$ at the same spatial frequencies for each value of time, and inverse spatial transform the product to produce the impulse response.

CASE II: MEDIA WITH LINEAR FREQUENCY DEPENDENCE OF LOSS COEFFICIENT

The wave equation is modelled by the telegrapher's equation of the form,

$$\nabla^2\phi - \frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} - A\frac{\partial\phi}{\partial t} = 0 \tag{9}$$

This model was proposed by Leeman in Refs. 9 and 10. The model begins with a more complete expression that simplifies to Eq. 9 when one wishes to model only the attenuation behavior in tissue in the frequency region from 0.5 to 10 MHz. Leeman[10] asserts that while the model does not accurately predict the signal velocities measured in tissues without additional terms, it does agree with measured values of attenuation in tissues in the region from 0.5 to 10 MHz.

The Green's function for this case has been found and is given in Ref 10. For the purpose of finding the propagation transfer function, the spatial transform of the Green's function is required. This transform is found by an alternate method[12] to be:

$$\bar{g}(f_x, f_y, z, t) = I_0\left[\frac{(A^2c^4-4\rho^2)^{1/2}(c^2t^2-z^2)^{1/2}}{2}\right]\exp[-(A/2)c^2t]H(ct-z)$$

$$\text{for } Ac/2 > \rho \tag{10}$$

where I_0 is the modified Bessel function. For $(Ac/2) < \rho$, the argument of the Bessel function becomes imaginary and the modified Bessel function will become the conventional Bessell function J_0. By its definition, the propagation transfer function is equal to the transform of the Green's function, hence we have

$$\bar{h}_{p2} = I_0\left[\frac{(A^2c^4-4\rho^2)^{1/2}(c^2t^2-z^2)^{1/2}}{2}\right]\exp[-(A/2)c^2t]H(ct-z) \tag{11}$$

The transform of the spatial impulse response is

$$\bar{h}(f_x, f_y, z, t) = \bar{s}(f_x, f_y)\, I_0\left[\frac{(4\rho^2-A^2c^2)^{1/2}(c^2t^2-z^2)^{1/2}}{2}\right]\exp[-(A/2)c^2t]H(ct-z) \tag{12}$$

Finding the spatial impulse response requires taking the inverse spatial transform giving

$$h(x,y,z,t) = F^{-1}\left[\bar{s}(f_x, f_y)\, I_0\left[\frac{(4\rho^2-A^2c^2)^{1/2}(c^2t^2-z^2)^{1/2}}{2}\right]\exp[-(A/2)c^2t]\right]H(ct-z) \tag{13}$$

Hence evaluation of the spatial impulse response requires finding the transform of the known function $s(x,y)$, evaluation of the value of the propagation transfer function $\bar{h}_{p2}$ at each spatial frequency, and evaluating the inverse transform of the product.

CASE III: MEDIA WITH A QUADRATIC FREQUENCY DEPENDENCE OF LOSS COEFFICIENT

The wave equation for this case is[14] the Stoke's equation (or the 'modified' wave equation).

$$\nabla^2\phi - \frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} - \beta\frac{\partial\nabla^2\phi}{\partial t} = 0 \tag{14}$$

where c is the nominal acoustic velocity and β is the absorption coefficient. For typical liquids or gases, this equation will provide an absorption coefficient that varies as the square of the frequency[14]. To simplify the math, we will require that the source is axisymmetric. Since the Helmholtz-Kirchhoff radiation integral is not valid for this equation, we need to begin by finding a radiation integral. Using distribution theory it can be shown[13] that the radiation integral that applies to this case is

$$\phi(r,t) = \int_S \left[\phi(r_0,t) \underset{t}{*} \frac{\partial g(r-r_0,t)}{\partial n} - g(r-r_0,t) \underset{t}{*} \frac{\partial\phi(r_0,t)}{\partial n}\right] dS$$

$$+ \beta\frac{\partial}{\partial t}\left[\int_S \left[\phi(r_0,t) \underset{t}{*} t\frac{\partial g(r-r_0,t)}{\partial n} - g(r-r_0,t) \underset{t}{*} \frac{\partial\phi(r_0,t)}{\partial n}\right] dS\right] \tag{15}$$

This equation shows how the field is related to the value of the function and its normal derivative on the surface S enclosing the volume V. The next step is to find an elementary solution to the wave equation. We choose a point source of the form

$$p(x,y,z,t) = \delta(t-t_0)\delta(x-x_0)\delta(y-y_0)\delta(z-z_0) \tag{16}$$

By assuming a typical value of β of 10^{-10} and restricting the frequency to one that satisfies the inequality

$$\beta\omega^2 t \leq 100 \tag{17}$$

one can show[13] that the outward-travelling elementary solution to the wave equation, $e(x,y,z,t)$, is

$$e(x,y,z,t) = \frac{\exp\left[-\dfrac{(ct-R)^2}{2c^2\beta t}\right]}{Rc(\beta t)^{1/2}} \tag{18}$$

The next step is to impose the rigid baffle boundary condition in order to find the Green's function. Subject to the same assumptions made in finding the elementary solution, we can show that the Green's function of the wave equation can be approximated[13] by

$$g(x,y,z,t) = \frac{2\exp\left[-\dfrac{[ct-(r^2+z^2)^{1/2}]^2}{2c^2\beta t}\right]}{c(\beta t)^{1/2}[r^2+z^2]^{1/2}} \tag{19}$$

where $r=(x^2+y^2)^{1/2}$. This Green's function solves the wave equations (subject to the approximations discussed) and meets the boundary condition that the normal derivative is zero at z=0. The next step is to find the spatial impulse response. Substituting the separable source expression into the radiation integral of Eq. 15, we find

$$h(x,y,z,t) = s(x,y) \underset{xy}{**} \left[g(x,y,z,t) + \beta \frac{\partial g}{\partial t} \right] \tag{20}$$

Taking the two-dimensional Fourier transform of Eq. 20 gives

$$\bar{h} = \bar{s} \left[\bar{g} + \beta \frac{\partial \bar{g}}{\partial t} \right] \tag{21}$$

Equation 21 relates the spatial spectrum of the impulse response to the spatial spectrum of the input velocity distribution. The term in the bracket is the transfer function $\bar{h}_{p3}$ of the propagation in the lossy medium. It represents a time-varying spatial filter that modifies the spatial spectrum of the input wave as both z and/or t change. The propagation transfer function is the term in the bracket in Eq. 21. For typical values of β that are on the order of 10^{-10} the second term of Eq. 21 will be negligible and

$$\bar{h} \simeq \bar{s}\bar{g} \tag{22}$$

where the transfer function for propagation $\bar{h}_{p3}$ in this medium is seen to be approximately $\bar{g}$. Finding this transfer function by taking the two-dimensional spatial transform of the Green's function, we have

$$\bar{h}_{p3} = \exp[-\, 2\pi^2 c^2 \beta \rho^2 t] \times \left[\frac{\exp[-z^2/2c^2\beta t]}{c(\beta t)^{1/2}} \underset{z}{*} J_0[2\pi\rho(c^2t^2 - z^2)^{1/2}]H(ct-z) \right] \tag{23}$$

The method of finding the (approximate) impulse response is to take the two-dimensional spatial transform of the known spatial function s(x,y), evaluate the value of $\bar{h}_{p3}$ for each spatial frequency by using Eq. 23, and then take the inverse transform of the product. We note that the term on the right side of the convolution in Eq. 23 is the same propagation transfer function associated with propagation through a lossless medium.

NUMERICAL SIMULATIONS

The following calculations have been done using a 64x64 array of data for 64 points in time. While the method gives a three-dimensional solution at any given observation distance, one dimension is eliminated in the plots by representing the solution through a median of the source, as is conventionally done in the literature. The plots show the amplitude of the wave plotted against cross-direction and time. For plotting convenience, the plots have been normalized to the maximum amplitude value obtained for lossless propagation. The width is normalized to the characteristic source size, D, (i.e., either the diameter or the width), and the time axis is normalized by the value of D/c. The origin of the time axis begins at z/c, the instant that the first part of the wave arrives at the observation plane. All plots are in an observation plane located 10 cm in front of the source plane.

Figures 2–4 show the calculated impulse response from a square piston source (i.e., s(x,y) is a uniform square). The values of the loss coeffi-

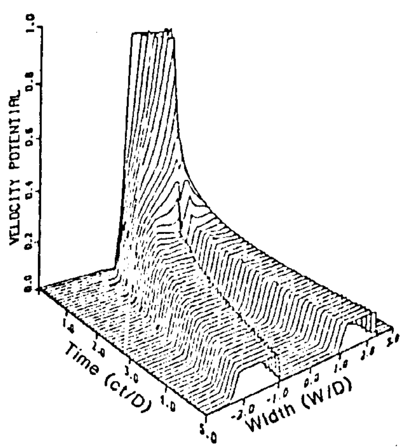

Fig. 2 Square transducer, impulse excitation, z=10 cm, lossless diffraction, D=2.2 cm

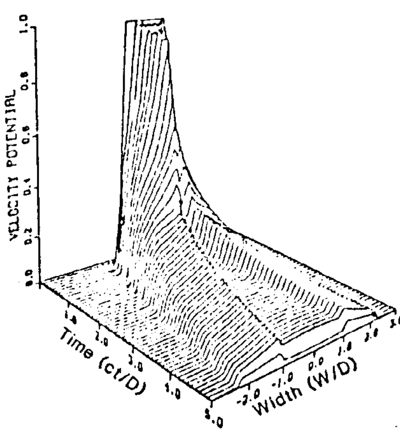

Fig. 3 Square transducer, impulse excitation, z=10 cm, lossy medium ($A = 1.5\times10^{-3}$ s-m^{-2}), D=2.2 cm

cient in the lossy media are given in the captions. The lossy media are seen to attenuate the waves and to cause a filling in of the region between the 'tails' of the wave as time proceeds. Also the lower spatial frequencies are seen to increasingly dominate as time increases.

To illustrate a spatially nonuniform excitation, we consider a circular region (diameter is D) with a Gaussian spatial excitation. The 1/e widths are indicated in the captions. The calculated impulse responses are shown in Figs. 5–7. The shape of the Gaussian wave stays much the same in

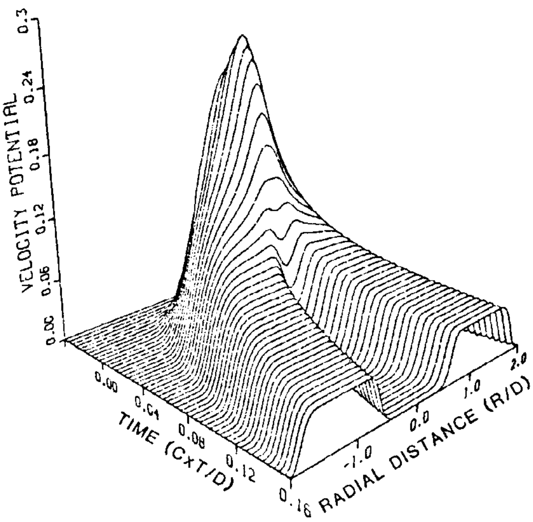

Fig. 4 Square transducer, impulse excitation, z=10 cm, lossy medium ($\beta = 10^{-9}$ s), D=3.1 cm

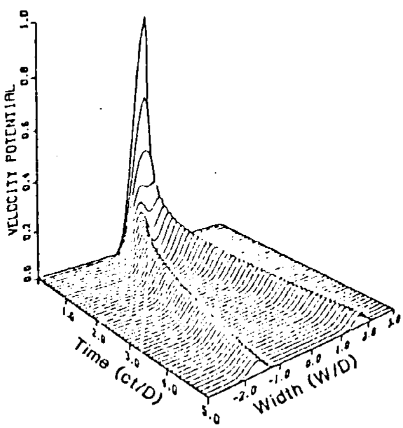

Fig. 5 Circular source, Gaussian spatial excitation, z=10 cm lossless medium (A = 0), 1/e point = 0.491 cm, D= 2.2 cm

both the low loss and high loss cases because of the lower spatial frequency content of this waveshape.

For a time excitation different than $\delta(t)$, the diffracted wave is a convolution between the impulse response and the time derivative of the temporal excitation portion of the source acoustic potential given by Eq. 1. The pressure of the wave is proportional to the time derivative of the acoustic velocity potential. Figure 8 shows the pressure pattern from a uniformly excited square velocity with a one-cycle square wave temporal excitation. The period of the square wave is 8×10^{-3} D/c. The effects of the time derivative are noticeable along the time axis.

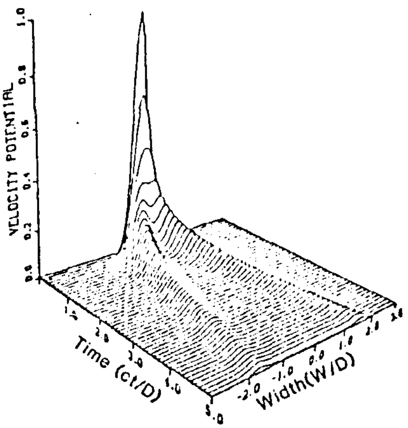

Fig. 6 Circular source, Gaussian spatial excitation, z=10 cm lossy medium ($A = 1.5x10^{-3}$ $s\text{-}m^{-2}$), 1/e point = 0.491 cm, D=2.2 cm

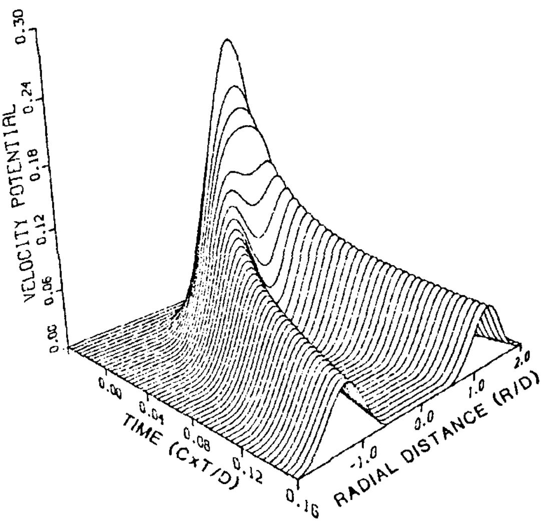

Fig. 7 Circular source, Gaussian spatial excitation, z=10 cm lossy medium ($\beta = 10^{-9}$ s), 1/e point = 1.18 cm, D = 3.1 cm

SUMMARY

This paper presents a computationally efficient method of computing the transient acoustic waves in lossless and lossy media. The fields are expressed in terms of the spatial impulse response which is found by inverse transforming the product of the transform of the spatial excitation and the appropriate propagation transfer function for the medium. No geometrical interpretations are required as the method uses only the spatial Fourier transform (or Hankel transform) in its computations.

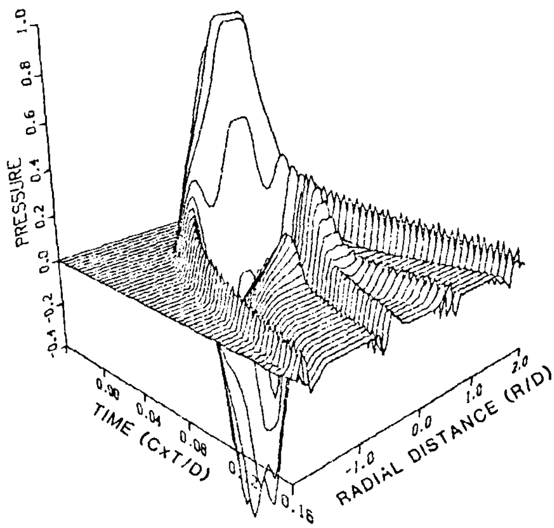

Fig. 8 Transient pressure response from square transducer with a single-cycle square wave excitation with a period of 8×10^{-3} D/c (z=10 cm, D=3.1 cm, $\beta=10^{-10}$ s)

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